S.-T. Yau College Student Mathematics Contests 2025

Analysis and Differential Equations

(6 problems)

Problem 1.

Find the solution $R(r)$ $(r \geq 1)$ of the differential equation

$$r^3 R'''(r) + 2r^2 R''(r) - rR'(r) + R(r) = 2025, \quad r > 1,$$

satisfying the conditions

$$R(1) = 2025, \quad R'(1) = 0 \quad \text{and} \quad R''(1) = 1.$$

Problem 2.

(1). Prove that for $n \geq 3$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}\, dx \leq C \int_{\mathbb{R}^n} |\nabla u|^2\, dx, \quad \forall u \in H^1(\mathbb{R}^n);$$

(2). Prove that the optimal constant C in (1) is $\frac{4}{(n-2)^2}$.

Problem 3.

(1). Let $u(x, y)$ be a subharmonic function on a domain D containing the annulus $\{r_1 < |x| < r_2\}$. Define

$$M(r) = \max_{x^2+y^2=r} u(x, y)$$

for $r_1 < r < r_2$. Prove that:

$$M(r) \leq \frac{M(r_1)(\log r_2 - \log r) + M(r_2)(\log r - \log r_1)}{\log r_2 - \log r_1};$$

(2). If u is subharmonic on $\mathbb{R}^2 \setminus \{0\}$ and bounded above, prove that u is constant.

Problem 4.

Write $\mathbf{S}^1 = \{e^{2\pi\sqrt{-1}s} : s \in \mathbf{R}\} \cong \{s : s \in [0, 1]\} = [0, 1]$. For a *smooth closed curve*

$$\gamma : \mathbf{S}^1 \longrightarrow \mathbf{R}^2, \quad s \longmapsto \gamma(s) = (x(s), y(s))$$

we mean that $x, y : \mathbf{S}^1 \to \mathbf{R}$ are continuously differentiable functions of s, $\gamma(0) = \gamma(1)$, and $\gamma'(s) = \dfrac{d\gamma(s)}{ds} \neq 0$ for all $s \in [0, 1]$.

The *degree* of a smooth closed curve $\gamma : \mathbf{S}^1 \to \mathbf{R}^2$ is given by

$$d(\gamma) := \frac{1}{2\pi} \oint_\gamma \frac{xdy - ydx}{x^2 + y^2}.$$

(1) Show that

$$d(\gamma) = \frac{1}{2\pi\sqrt{-1}} \oint_\gamma \frac{dz}{z}.$$

Hence $d(\gamma)$ is an integer.

(2) For the curve $\gamma(s) = re^{2\pi\sqrt{-1}ns} \in \mathbf{C}$ with $z_0 \in \mathbf{C}$, $s \in [0,1]$ and $n \in \mathbf{Z}$, compute $d(\gamma)$.

(3) Two smooth closed curves γ_0 and γ_1 are *regularly homotopic* if there exists a family of smooth closed curves γ_t with a continuous dependence on $t \in [0,1]$, where the curves γ_0 and γ_1 correspond to $t = 0$ and $t = 1$ respectively. Here, by a continuous dependence on t we mean that the map

$$[0,1] \times [0,1] \longrightarrow \mathbf{R}^2, \quad (s,t) \longmapsto \gamma(s,t) = \gamma_t(s)$$

is continuous.

Prove that two smooth closed curves γ_0 and γ_1 are regularly homotopic if and only if there have equal degrees.

Problem 5.
If A and B are disjoint closed subsets of a metric space (X,d) and $[a,b]$ is a given closed interval, then there exists a continuous map $f : X \to [a,b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.

Problem 6.
For f a complex-valued locally integrable function on $\mathbf{R}^n$, set

$$||f||_{\mathbf{BMO}} := \sup_Q \left(\frac{1}{|Q|} \int_Q \left| f(x) - \mathbf{Avg}_Q f \right| dx \right),$$

where the supremum is taken over all cubes Q in $\mathbf{R}^n$ and

$$\mathbf{Avg}_Q f := \frac{1}{|Q|} \int_Q f(x)\, dx.$$

Let $\mathbf{BMO}(\mathbf{R}^n)$ be the set of all complex-valued locally integrable functions on $\mathbf{R}^n$ with $||f||_{\mathbf{BMO}} < \infty$.

(1) $||\cdot||_{\mathbf{BMO}}$ is not a norm, and is only a seminorm.

(2) $L^\infty(\mathbf{R}^n) \subset \mathbf{BMO}(\mathbf{R}^n)$ and $||f||_{\mathbf{BMO}} \le 2||f||_{L^\infty}$.

(3) Suppose that there exists an $A > 0$ such that for all cubes Q in $\mathbf{R}^n$ there exists a constant c_Q such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - c_Q|\, dx \right) \le A.$$

then $f \in \mathbf{BMO}(\mathbf{R}^n)$ and $||f||_{\mathbf{BMO}} \le 2A$.

(4) $L^\infty(\mathbf{R}^n)$ is a proper subspace of $\mathbf{BMO}(\mathbf{R}^n)$.